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# STRUCTURAL DESIGN USING EQUILIBRIUM PROGRAMMING

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## Abstract

Equilibrium programming is investigated as a framework within which optimal structural design problems can be decomposed into subproblems. Aspects of equilibrium programming theory pertinent to this application are described, and three decompositions of a structural design problem into structural-sizing and structural-response subproblems are developed. Two of the decompositions are shown to be equivalent to existing solution methods. However, the third method is novel and is shown to give optimal solutions both theoretically, and in two numerical examples.

## Introduction

Nonlinear mathematical programming (NLP) is extensively used for optimal structural sizing in structural design. Utilizing NLP, a sizing design problem minimizes an objective function, such as weight, and satisfies a set of design constraints, such as minimum gauge and local stress limits, by varying a set of structural-sizing design variables, such as facesheet thicknesses, and stiffener heights. The structural analysis that is required to compute the design constraints is subordinate to the optimization algorithm in the simplest use of NLP. That is, the structural response is recalculated for every change in the values of structural-sizing design variables in the design process. The disadvantage of this approach is the expense of performing numerous structural-response and sensitivity-derivative calculations during the design process. The incorporation of approximation concepts by Schmit and Miura (ref. 1) within NLP structural-sizing design methods significantly improved the computational efficiency of the method. Thus, approximation-based NLP structural-sizing design methods are now well developed, and are implemented in several commercial structural analysis codes.

Though approximation-based NLP structural-sizing design methods have a computational advantage over simple NLP approaches, the design of structures incorporating a high degree of detail requires a large number of design variables and constraints. Additional improvements in optimal structural-sizing methodology are required to deal with increasingly large numbers of design variables and constraints. One approach to the solution of large problems is to decompose a large problem into

a set of smaller subproblems. Most of the smaller subproblems can be solved separately and in parallel to reduce the time required to find the optimum design. By repeating this decomposition process at the subproblem level, a multilevel decomposition (ref. 2 and references therein) can be obtained. One concern in the decomposition of an optimum design problem is whether any solution exists to the decomposed problem. If a solution exists, another concern is whether this solution to the decomposed problem is also a solution to the original problem. Although progress has been made in addressing these concerns (see ref. 3), and in utilizing multilevel decomposition for optimal design, a mathematical theory that provides a framework for performing design decomposition appears lacking.

The theory of equilibrium programming (EP) provides a framework to analyze multiple, interdependent nonlinear programming problems. Equilibrium programming was developed in an operations research setting, and has been applied to economics, game theory, and network theory (ref. 4). Approaches to decomposition of structural-sizing design problems utilizing an equilibrium programming framework are developed in the present study. The decompositions examined investigate several ways to separate the design problem into a structural-sizing subproblem, and one or more structural-response subproblems. The theorems on the existence of, and the necessary conditions for an EP solution are utilized to develop decompositions that have solutions, and to determine whether a solution is optimal. In subsequent sections of the present paper, the characteristics of EP solutions are described, and some EP-based approaches to decomposition in structural design are summarized. Specifically, structural-response and structural-sizing problems for minimum weight design in the presence of structural-sizing design variable constraints, local buckling, stress, and displacement constraints are presented, and decomposed into several formulations utilizing EP. It is shown herein that two established structural-design methods; namely, fully stressed design (ref. 5) and NLP-based structural sizing using first-order Taylor series approximations of the constraints (ref. 1), can be derived using EP. A new decomposition derived using the EP framework is also presented. Computational results from this new EP-based formulation are compared with results from an established design method.

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## Nomenclature

Lower case symbols typed in boldface denote column vectors, however some vectors are not denoted in this manner. The notation  $\partial f/\partial \mathbf{a}$  for an arbitrary scalar  $f$  and a vector  $\mathbf{a} = (a_1, \dots, a_n)^T$  denotes the gradient of scalar  $f$  with respect to vector  $\mathbf{a}$ . This gradient is expressed as a row vector (i.e.,  $\partial f/\partial \mathbf{a} = (\partial f/\partial a_1, \dots, \partial f/\partial a_n)$ ). The notation  $\partial \mathbf{f}/\partial \mathbf{a}$  for arbitrary vectors  $\mathbf{f} = (f_1, \dots, f_m)^T$  and  $\mathbf{a} = (a_1, \dots, a_n)^T$  denotes the gradient of vector  $\mathbf{f}$  with respect to vector  $\mathbf{a}$ . This gradient is expressed as an  $m \times n$  matrix in which the component in column  $j$  of row  $i$  is given by  $\partial f_i/\partial a_j$ . The notation  $d\mathbf{f}/d\mathbf{a}$  is utilized to represent numerical values of the total derivative of  $\mathbf{f}$  with respect to the vector  $\mathbf{a}$  using a matrix format similar to that of  $\partial \mathbf{f}/\partial \mathbf{a}$ .

<b>A</b>	vector of cross-sectional areas
<b>g</b>	vector of inequality constraint functions
<b>F<sub>E</sub></b>	vector of external nodal forces
<b>f</b>	objective function in nonlinear programming
<b>h</b>	vector of equality constraint functions
<b>K</b>	global stiffness matrix for a structural-response problem
<b>L</b>	Lagrangian function, (see equation (20))
<b>M</b>	number of structural-response cases
<b>N</b>	vector of stress resultants within every element of the structure
<b>u</b>	vector of nodal displacements from a structural-response problem
<b>v</b>	vector of design variables for nonlinear programming, or vector of structural-sizing design variables in equilibrium programming structural-sizing subproblems
<b>V</b>	strain energy of a structure
<b>W</b>	weight of a structure
<b>x, x<sup>i</sup>, <math>\bar{x}^i</math></b>	vectors of all design variables for equilibrium programming, design variables of subproblem $i$ , and all design variables except those from subproblem $i$ , respectively
<b><math>\lambda</math></b>	vector of Lagrange multipliers for inequality constraints
<b><math>\mu</math></b>	vector of Lagrange multipliers for equality constraints, (used for inequality constraints in the appendix)

**$\sigma$**  vector of stresses within every element of a structure

Subscripts and Subscripts:

<b>A</b>	denotes an approximation of a function
<b>i</b>	denotes $i$ 'th equilibrium programming subproblem, or $i$ 'th load condition for structural response
<b>max</b>	maximum
<b>min</b>	minimum
<b>N</b>	denotes functional dependence on stress resultants
<b>s</b>	side constraints
<b>T</b>	transpose of a vector or matrix
<b>u</b>	denotes displacement constraints, or functional dependence on displacements
<b><math>\sigma</math></b>	denotes stress and local buckling constraints, or functional dependence on stresses
<b>*</b>	denotes an equilibrium programming solution (i.e., the equilibrium point), an optimal solution, active constraints, or Lagrange multipliers corresponding to active constraints

## Theory

To have a basis for comparison with the equilibrium programming (EP)-based formulations that follow, this section begins with a summary of the methodology for utilizing nonlinear programming (NLP) for structural design. Some results from EP theory are then presented, and some characteristics of equilibrium programming problems are described. Three decompositions of structural-design problems are developed using the EP framework, and some implementation issues are addressed.

### Nonlinear Programming Approach to Structural Design

Nonlinear programming can be personified as having one decision maker (which may be implemented as a search algorithm) with control of a set of design variables given by the vector  $\mathbf{v}$ . The goal of the decision maker is to minimize an objective function  $f(\mathbf{v})$  while satisfying a set of constraints. The mathematical description of a NLP problem is:

$$\begin{aligned} \min_{\mathbf{v}} \quad & f(\mathbf{v}) \\ \mathbf{g}(\mathbf{v}) \geq \quad & 0 \\ \mathbf{h}(\mathbf{v}) = \quad & 0 \end{aligned} \tag{1}$$

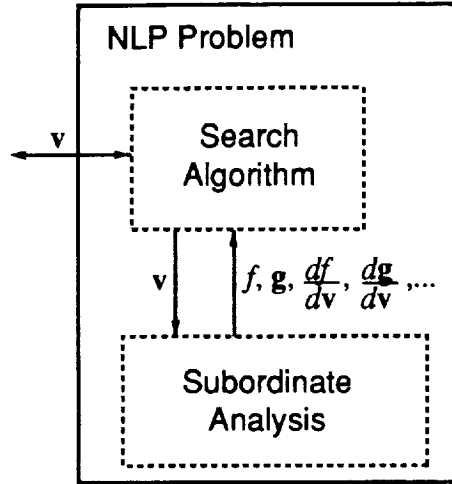


Figure 1. Block diagram illustrating the major components of a solution algorithm for a NLP problem.

where  $g(v)$  are inequality constraints – which could include simple bounds on the design variables – and  $h(v)$  are equality constraints. A simple block diagram illustrating the major components of a solution algorithm for a NLP problem is shown in figure 1. As shown in this figure, the analysis module is subordinate to the algorithm that searches the design space for the optimum design variables, and it can be queried for function (and gradient) values at will.

The first-order conditions that are necessary for  $v = v^*$  to be a solution to the NLP problem represented by statement (1) are often used in solution procedures. If a constraint qualification – such as linear independence of the active constraints at  $v = v^*$  – is satisfied, then a set of Lagrange multipliers  $(\lambda, \mu)$  exists such that at  $v = v^*$  the following necessary condition relations are satisfied:

$$\begin{aligned} \frac{\partial f(v)}{\partial v} - \lambda^T \frac{\partial g(v)}{\partial v} - \mu^T \frac{\partial h(v)}{\partial v} &= 0 \\ g(v) &\geq 0 \\ h(v) &= 0 \\ \lambda &\geq 0 \\ \lambda^T g(v) &= 0 \end{aligned} \quad (2)$$

The functions  $g(v)$  and  $h(v)$  for a structural-design problem implicitly involve the structural response; so results from structural analyses are necessary to determine these constraint functions.

**Structural Analysis.** A minimum potential energy formulation can be used to calculate the structural displacements and internal stresses that are required to evaluate a design. Given the structural arrangement,

sizes for all the structural elements, a discretization of the structure into finite elements, and a set of external forces on the discretized structure  $F_E$ , the correct structural displacements are those that minimize the potential energy of the structure. Thus, the structural response is the solution to the unconstrained NLP problem given by

$$\min_u (V(u) - F_E^T u) \quad (3)$$

where  $u$  is a vector of nodal displacements,  $V(u)$  is the strain energy of the structure, and  $F_E$  is a vector of external nodal forces. In the present study, a linear structural analysis is assumed, the strain energy is given by  $V(u) = 1/2 u^T K u$ , and the necessary condition relations (2) for the unconstrained minimization problem represented by statement (3) simplify to the linear equations:

$$K u = F_E \quad (4)$$

where  $K$  is the global stiffness matrix. Once the displacements  $u$  have been determined from equation (4), stresses  $\sigma$  within every element of the structure can be calculated from the displacements, the element strain-displacement relations, and the element elastic constants.

**Optimal Structural Sizing.** A common objective function for structural sizing is the weight of the structure  $W(v)$ . The structural-sizing design variables  $v$ , referred to herein as sizing variables, can be the dimensions of the individual elements that explicitly contribute to weight, such as beam dimensions, facesheet thicknesses, stiffener dimensions and spacing, and the thicknesses of a non-load-bearing spacer material; or they can be factors which effect the weight in an indirect manner, such as the orientation of composite fibers. The constraint functions considered in the present study are the sizing variable constraints, or side constraints,  $g_s(v)$ ; the local buckling and stress constraints  $g_\sigma^i(v, \sigma^i(v))$ ; and the displacement constraints  $g_u^i(u^i(v))$ . The superscript  $i$  indicates that these quantities correspond to structural load condition  $i$  (i.e., to the vector of external nodal forces  $F_E^i$ ). The displacement constraint functions  $g_u^i(u^i(v))$  are assumed to have no direct dependence on the sizing variables  $v$ , but the functional form  $u^i(v)$  indicates an indirect dependence on  $v$  through the structural analysis. The stresses  $\sigma(v)$ , which are shown to depend on  $v$  in the constraint functions, can have several forms. For example, one can write  $\sigma(v) = \sigma_u(v, u(v)) = \sigma_N(v, N(v, u(v)))$  to show that the functional form for stresses can depend directly on displacements  $u$ , or indirectly on displacements through the stress resultants  $N$ . Very often the side constraints, the displacement constraints, and the stress constraints reduce to simple bounds on the sizing

variables, the displacements, and the stresses, respectively. Thus, the NLP-based design approach in which the structural analyses are treated as subordinate to the sizing variable search algorithm can be summarized by:

$$\begin{aligned} \min_{\mathbf{v}} W(\mathbf{v}) \\ \mathbf{g}_s(\mathbf{v}) \geq 0 \\ \mathbf{g}_u^i(\mathbf{u}^i(\mathbf{v})) \geq 0 \\ \mathbf{g}_\sigma^i(\mathbf{v}, \boldsymbol{\sigma}^i(\mathbf{v})) \geq 0 \end{aligned} \quad (5)$$

where  $i = 1, \dots, M$  for  $M$  load conditions. The necessary conditions for  $\mathbf{v} = \mathbf{v}^*$  to be the solution to the NLP structural-design problem given above are as follows:

$$\begin{aligned} \frac{\partial W(\mathbf{v})}{\partial \mathbf{v}} - \lambda_s^T \frac{\partial \mathbf{g}_s(\mathbf{v})}{\partial \mathbf{v}} \\ - \sum_{k=1}^M \left[ (\lambda_u^k)^T \frac{\partial \mathbf{g}_u^k(\mathbf{u}^k)}{\partial \mathbf{u}^k} \frac{\partial \mathbf{u}^k(\mathbf{v})}{\partial \mathbf{v}} + (\lambda_\sigma^k)^T \right. \\ \left. \times \left( \frac{\partial \mathbf{g}_\sigma^k(\mathbf{v}, \boldsymbol{\sigma}^k)}{\partial \mathbf{v}} + \frac{\partial \mathbf{g}_\sigma^k(\mathbf{v}, \boldsymbol{\sigma}^k)}{\partial \boldsymbol{\sigma}^k} \frac{\partial \boldsymbol{\sigma}^k(\mathbf{v})}{\partial \mathbf{v}} \right) \right] = 0 \\ \mathbf{g}_s(\mathbf{v}) \geq 0 \\ \mathbf{g}_u^i(\mathbf{u}^i(\mathbf{v})) \geq 0 \\ \mathbf{g}_\sigma^i(\mathbf{v}, \boldsymbol{\sigma}^i(\mathbf{v})) \geq 0 \\ \lambda_s \geq 0 \\ \lambda_u^i \geq 0 \\ \lambda_\sigma^i \geq 0 \\ (\lambda_s)^T \mathbf{g}_s(\mathbf{v}) = 0 \\ (\lambda_u^i)^T \mathbf{g}_u^i(\mathbf{u}^i(\mathbf{v})) = 0 \\ (\lambda_\sigma^i)^T \mathbf{g}_\sigma^i(\mathbf{v}, \boldsymbol{\sigma}^i(\mathbf{v})) = 0 \end{aligned} \quad (6)$$

where  $i = 1, \dots, M$ . The terms  $\mathbf{u}^i(\mathbf{v})$  and  $\partial \mathbf{u}^i(\mathbf{v})/\partial \mathbf{v}$  are found by solving the following structural-analysis and sensitivity-derivative equations in which the parameter vector  $\mathbf{v}$  is assumed known:

$$\begin{aligned} \mathbf{K}(\mathbf{v})\mathbf{u}^i - \mathbf{F}_E^i &= 0 \\ \mathbf{K}(\mathbf{v}) \frac{\partial \mathbf{u}^i}{\partial \mathbf{v}} + \frac{\partial \mathbf{K}(\mathbf{v})\mathbf{u}^i}{\partial \mathbf{v}} &= [0] \end{aligned} \quad (7)$$

where it is also assumed that the nodal forces  $\mathbf{F}_E^i$  are independent of  $\mathbf{v}$ .

### Equilibrium Programming Theory

In this section of the present paper, the formulation of an equilibrium programming problem is given. In addition, the first-order necessary conditions that are satisfied by a solution to an EP problem, a theorem on solution existence, some solution properties, and some

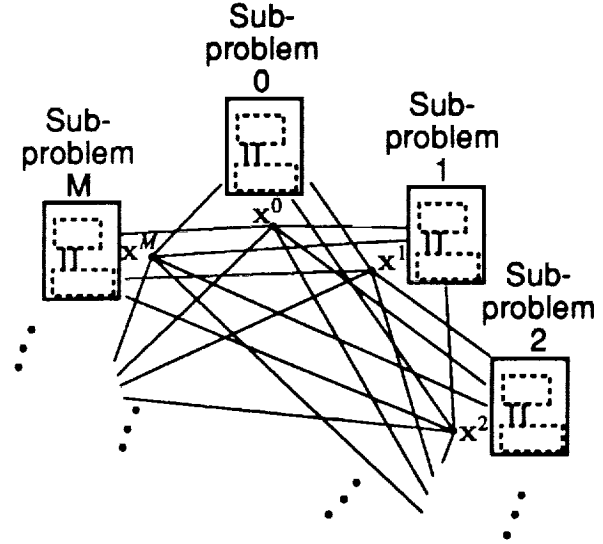


Figure 2. Block diagram illustrating the subproblem coupling in an EP problem.

solution algorithms are summarized. References are made to the application of EP to design decomposition where appropriate.

Equilibrium programming is a generalization of NLP which can be personified as having  $M + 1$  decision makers (which may be implemented as search algorithms) that interact in a system. Each decision maker has a NLP subproblem to solve, and an independent set of design variables to control. To simplify notation, the design variables controlled by decision maker  $i$  are denoted as  $\mathbf{x}^i$ , the design variables of all  $M + 1$  decision makers are denoted as  $\mathbf{x} = (\mathbf{x}^0, \dots, \mathbf{x}^M)$ , and all design variables not controlled by decision maker  $i$  are denoted as  $\bar{\mathbf{x}}^i = (\mathbf{x}^0, \dots, \mathbf{x}^{i-1}, \mathbf{x}^{i+1}, \dots, \mathbf{x}^M)$ . Decision maker  $i$  has an objective function to minimize,  $f^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)$ , while satisfying a set of constraints. Thus, the mathematical description of equilibrium programming is:

$$\begin{aligned} \min_{\mathbf{x}^i} f^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) \\ \mathbf{g}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) \geq 0 \\ \mathbf{h}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) = 0 \end{aligned} \quad (8)$$

for the  $i = 0, \dots, M$  interacting subproblems. The variables following the comma in any of the functions in statement (8) are treated as fixed parameters in that subproblem. Thus, in the NLP problem of decision maker  $i$ , the design variables from other decision makers  $\bar{\mathbf{x}}^i$  enter as parameters. The coupling of the  $M + 1$  NLP subproblems through their design variables is illustrated in the block diagram of figure 2. In this figure, each subproblem is a NLP problem which might be solved by a solution algorithm like that of figure 1.

The solution to all the NLP subproblems represented by statement (8) is  $\mathbf{x} = \mathbf{x}^*$ , which is called an *equilibrium point*. In a manner similar to NLP, first-order necessary conditions are satisfied at the equilibrium point subject to a constraint qualification. Thus, at the equilibrium point  $\mathbf{x} = \mathbf{x}^*$ , there exist Lagrange multipliers  $(\lambda^i, \mu^i)$  such that the following conditions are satisfied:

$$\begin{aligned} \frac{\partial f^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)}{\partial \mathbf{x}^i} - (\lambda^i)^T \frac{\partial \mathbf{g}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)}{\partial \mathbf{x}^i} \\ - (\mu^i)^T \frac{\partial \mathbf{h}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)}{\partial \mathbf{x}^i} &= 0 \\ \mathbf{g}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) &\geq 0 \\ \mathbf{h}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) &= 0 \\ \lambda^i &\geq 0 \\ (\lambda^i)^T \mathbf{g}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) &= 0 \end{aligned} \quad (9)$$

for  $i = 0, \dots, M$ .

**Existence and Optimality of an Equilibrium Point.** In utilizing EP for a design decomposition, it is important that a solution exists for the decomposed problem. In fact, the requirements for existence of an equilibrium point can be used as a guide in the decomposition process. A second concern involves the optimality of the equilibrium point. That is, is the equilibrium point resulting from the decomposition of a design problem also an optimal solution of the original NLP formulation? This second concern can be addressed by showing that EP necessary condition relations (9) for the decomposed problem imply the NLP necessary condition relations (6) of the undecomposed problem. Both the existence and the optimality of an equilibrium point depend on the satisfaction of a constraint qualification.

The satisfaction of a constraint qualification is required for both the existence of a solution to the first-order conditions represented by statement (9), and the existence of an equilibrium point. One form for the constraint qualification is given in reference 6. It is satisfied if for all  $\mathbf{x}$  feasible to the NLP subproblems represented by statement (8), and for every  $i$ : 1) the vectors  $\partial \mathbf{h}_j^i(\mathbf{x}^i, \bar{\mathbf{x}}^i) / \partial \mathbf{x}^i$  are linearly independent, and 2) there is at least one solution  $\mathbf{z}^i$  to the relations

$$\begin{aligned} \frac{\partial \mathbf{g}_*^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)}{\partial \mathbf{x}^i} \mathbf{z}^i &> 0 \\ \frac{\partial \mathbf{h}^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)}{\partial \mathbf{x}^i} \mathbf{z}^i &= 0 \end{aligned} \quad (10)$$

where  $\mathbf{g}_*^i$  is the vector of inequality constraints in subproblem  $i$  that are active at  $\mathbf{x}$ . With regard to the inequality constraints, this constraint qualification essentially states that it is always possible to move into the

interior of the feasible region from a point on the boundary of that region. However, because constraint qualification relations (10) must be satisfied individually by the EP subproblems, the constraint functions in EP must satisfy more restrictive requirements than those in NLP. For example, the constraints from two EP subproblems given by  $\mathbf{g}_*^1(\cdot, \mathbf{x}^2)$  and  $\mathbf{g}_*^2(\mathbf{x}^1, \cdot)$  will not satisfy relation (10) individually because  $(\partial \mathbf{g}_*^1 / \partial \mathbf{x}^1) \mathbf{z}^1 \equiv 0$ . However, relation (10) may be satisfied for this example when the constraints and design variables of the subproblems are combined within a single NLP problem (i.e.,  $(\partial \mathbf{g}_* / \partial \mathbf{x}) \mathbf{z} > 0$  where  $\mathbf{g}_* = (\mathbf{g}_*^1, \mathbf{g}_*^2)$  and  $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ ).

A very general theorem for existence of an equilibrium point, given in reference 6, requires continuity, but not differentiability, of the objective and constraint functions. Other conditions for existence are: 1) the functions satisfy constraint qualification relations (10) (actually only a weakened form of the constraint qualification is required), 2) the feasible region is bounded, 3) the functions  $f^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)$  and  $g_j^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)$  are concave in  $\mathbf{x}^i$ , and 4) the functions  $h_j^i(\mathbf{x}^i, \bar{\mathbf{x}}^i)$  are linear in  $\mathbf{x}^i$ . Other existence theorems are available which relax the concavity and linearity restrictions on  $(f^i, \mathbf{g}^i)$  and  $\mathbf{h}^i$ , respectively, if additional differentiability requirements are satisfied.

Three methods are used in the present study for the decomposition of a design problem into EP structuralizing and structural-response subproblems. The requirements for existence and optimality of the EP solutions are utilized as guidelines for applying these methods in the decomposition process. In the first decomposition method, a constraint is applied only within a subproblem for which the design variables are effective in satisfying the constraint. Here effectiveness is determined by ability to satisfy constraint qualification relations (10). This method is akin to the concurrent subspace optimization method for optimization decomposition described in reference 7. In the second decomposition method, a constraint is applied within a chosen subproblem, and a change of variables is performed to yield a set of the design variables and constraints which satisfy the constraint qualification relations. In the third decomposition method, a constraint is applied within a chosen subproblem, and parameters of the constraint functions which are normally independent of the subproblem design variables are replaced with approximate models that depend explicitly on the subproblem design variables, thus satisfying constraint qualification relations (10). For example, in a constraint function  $g_j^i(\bar{\mathbf{x}}^i)$  for subproblem  $i$ , the parameters  $\bar{\mathbf{x}}^i$  which are normally independent of the subproblem design variables  $\mathbf{x}^i$  are replaced with approximations that depend explicitly on  $\mathbf{x}^i$ . This method is closely related to the

approximation concepts of reference 1, and in fact, the use of explicit approximations of constraint functions in the NLP-based design methods can be viewed as a type of design decomposition. Examples of utilizing these decomposition methods in the development of structural-design equilibrium programming methods (SDEP) are given in a subsequent section of the present paper.

*Some Equilibrium Point Properties.* Although the differences between an equilibrium point and an optimal point may appear slight, they are important. The following equilibrium point properties, summarized from reference 4, illustrate the differences. Several examples of these differences can be found in the reference.

An equilibrium point is, in general, different from an optimal point (i.e., the solution of a NLP problem), even if the same constraints are satisfied and each EP subproblem has the same objective function. This difference can occur because the coupling of the constraint derivatives in the respective necessary conditions is generally less for an EP formulation than for a NLP formulation. Thus, the subproblems in EP can be called "loosely coupled." Larger differences between the EP and NLP solutions can occur when the  $M + 1$  subproblems in EP have  $M + 1$  different, and possibly conflicting, objective functions. (However, two of the structural-design decompositions described subsequently are specifically formulated to ensure that an equilibrium point is also optimal.) An equilibrium point has a "stability" property which states that a solution to the EP problem does not change for a perturbation of the design variables of a single subproblem from equilibrium values. Additional constraints can affect EP solutions differently than NLP solutions. In NLP, additional constraints generally increase the value of the objective function. However, in EP it is possible for additional constraints to force a "cooperation" that reduces the objective functions of all the EP subproblems.

*EP Solution Algorithms.* The solution of an EP problem can be obtained in several ways. The most straightforward method is to solve sequentially all the subproblems in some predetermined cyclical order. When the solutions to all the subproblems do not change from the previous iteration (within some numerical tolerance), the equilibrium point has been reached. This method is utilized in the present study. Other methods, such as solution of the set of nonlinear necessary condition equations (ref. 4), are also possible, and may be advantageous when convergence is difficult to achieve by the sequential solution method.

## Equilibrium Programming Approaches to Structural Design

In this section of the present paper, decompositions of a structural-design problem into structural-response

and structural-sizing subproblems are described. These decompositions are denoted as structural-design equilibrium programming (SDEP) formulations. In the succeeding formulations, it is assumed that there are  $M$  structural load conditions given by  $\mathbf{F}_E^i$ ,  $i = 1, \dots, M$ ; and that, in general, the constraints of interest are sizing variable, stress, local buckling, and displacement constraints. In some cases, the formulations simplify appreciably when these constraints are reduced to simple bounds. The formal definition of  $\mathbf{x}^i$  will be given for each subproblem. This formal definition may include both optimization design variables, and behavior variables which result from a post-processing analysis within the subproblem. The conversion of the analysis for behavior variables to subproblem equality constraints that conform to the EP formalism is obvious, and omitted in the development of the formulations.

The three SDEP formulations that follow are derived using the methods for decomposition described previously. The logical steps for utilizing the decomposition methods to derive the formulations are described separately for each formulation. The first structural-design equilibrium programming formulation developed considers only stress, local buckling, and minimum gauge constraints. This method, termed SDEP1, is shown to be equivalent to the method of fully stressed design for rod and membrane elements. The second and third SDEP formulations, denoted SDEP2 and SDEP3 respectively, provide for optimal designs with sizing variable, stress, local buckling, and displacement constraints. Formulation SDEP2 is shown to be equivalent to the NLP-based approach to optimal structural sizing using a first-order Taylor approximation for a rapid analysis. Formulation SDEP3 is new, and the existence and optimality of its equilibrium points are discussed.

*Structural-Design Formulation SDEP1.* It seems natural to define EP subproblems  $i = 1, \dots, M$ , corresponding to the  $M$  load conditions  $\mathbf{F}_E^i$ , to be structural-response subproblems, each a minimization problem represented by statement (3), and to define the subproblem design variables  $\mathbf{x}^i$  to be the displacements  $\mathbf{u}^i$  obtained from these minimization problems. This choice leaves subproblem 0 as the structural-sizing subproblem (i.e., minimize the structural weight by varying sizing variables  $\mathbf{x}^0 \equiv \mathbf{v}$  subject to the sizing variable, stress, and buckling limits in the NLP problem represented by statement (5)). The shortcoming of this EP formulation is that an equilibrium point may not exist because constraint qualification relations (10) cannot be satisfied for some constraints. For example, maximum stress constraints for rod elements having cross-sectional areas  $\mathbf{A}$  as design variables (i.e.,  $\mathbf{x}^0 \equiv \mathbf{v} \equiv \mathbf{A}$ ) are given by  $g_\sigma^i(\mathbf{x}) \equiv \sigma_{\max} - \sigma(\mathbf{u}^i) \geq 0$ . These constraints depend on the displacements  $\mathbf{x}^i \equiv \mathbf{u}^i$ , and not on  $\mathbf{x}^0 \equiv \mathbf{A}$ ; so there is no  $\mathbf{z}^0$  that will satisfy the inequality in relation



(10). Thus, a different decomposition is required.

The formulation SDEP1 utilizes the second decomposition method described previously to enable satisfaction of the constraint qualification relations. Since functions for calculating the stress and buckling constraints can be constructed by using both the sizing variables and the stress resultants of a structure, a change of variable is made to utilize stress resultants in the constraint functions. Because the stress and buckling constraint functions depend explicitly on the sizing variables with this change of variable, the satisfaction of the constraint qualification is much more likely, but satisfaction is not guaranteed as will be explained subsequently. The stress resultants within every element in the structure are computed within the structural-response subproblems, and are formally represented in subproblem  $i$  by the equation

$$\mathbf{N}^i = \mathbf{N}^i(\mathbf{v}, \mathbf{u}^i) \quad (11)$$

In formulation SDEP1, the structural-response subproblem  $i$  consists of a solution of the unconstrained minimization given by statements (3) or (4) for  $\mathbf{u}^i$  followed by calculation of stress resultants  $\mathbf{N}^i$  by equation (11). The design variables of structural-response subproblem  $i$  are defined as  $\mathbf{x}^i = (\mathbf{u}^i, \mathbf{N}^i)$ . The stress resultants  $\mathbf{N}^i$  are used instead of displacements in formulating the stress and buckling constraints of the structural-sizing subproblem, which is also denoted subproblem 0. Thus, the structural-sizing subproblem in equilibrium programming formulation SDEP1 is:

$$\begin{aligned} \min_{\mathbf{v}} \quad & W(\mathbf{v}) \\ \mathbf{v} - \mathbf{v}_{\min} \geq \quad & 0 \\ \mathbf{g}_{\sigma}^i(\mathbf{v}, \sigma_{\mathbf{N}}(\mathbf{v}, \mathbf{N}^i)) \geq \quad & 0 \end{aligned} \quad (12)$$

where  $i = 1, \dots, M$ . The design variables of this structural-sizing subproblem are simply  $\mathbf{x}^0 = \mathbf{v}$ .

The method of fully stressed design (ref. 5) can be derived from SDEP1 if: 1) only one-dimensional rod and two-dimensional membrane finite elements are used; 2) one sizing variable is associated with each finite element having a stress constraint; 3) the stress constraints limit the maximum stress magnitude or von Mises stress; and 4) there are no buckling constraints. The structural-sizing subproblem can then be decomposed into a set of independent elemental problems, one for each sizing variable and constrained element combination. The solution of these elemental problems is simple since the value of the sizing variable that makes a constraint active can be found analytically for each load case. For example, the elemental problem for cross-sectional area of one-dimensional rod element  $j$  is solved

by:

$$A_j = \max(A_{j,\min}, |P_{z,j}^{el1}|/\sigma_{j,\max}, \dots, |P_{z,j}^{elM}|/\sigma_{j,\max}) \quad (13)$$

where the function  $\max(\dots)$  chooses the maximum of its arguments, the structural-sizing design variable  $x_j^0$  is defined to be the sizing variable  $A_j$ , and  $P_{z,j}^{eli}$  is the axial force in the element  $j$  for load case  $i$ . The quantities  $P_{z,j}^{eli}$  are elements of the vectors  $\mathbf{N}^i$  which are also elements of  $\mathbf{x}^i$  and  $\bar{\mathbf{x}}^0$ . A solution method that alternates between solving the sizing elemental problems, and the structural-response subproblems leads to fully stressed design.

As stated previously, the change of variables that recasts the stress and buckling constraints in terms of sizing variables and stress resultants makes the satisfaction of the constraint qualification much more likely, but not guaranteed. A simple example makes this statement clear. Assume a rod is fixed between two rigid walls and its temperature is increased; the design problem is to size the rod cross-sectional area  $A$  to minimize weight and satisfy a maximum stress constraint. The temperature change induces a strain which can be expressed as an equivalent external load that is a function of the stiffness. Thus, the equivalent external load and the rod stress resultant are both calculated in the structural-response subproblem as  $P_z = k A \Delta T$  where  $k$  is a constant of proportionality, and  $\Delta T$  is the temperature rise. The stress, which is calculated in the structural-sizing subproblem, is given by  $\sigma = P_z/A$ , and also equals the quantity  $k \Delta T$ . If this quantity is larger than the stress limit, it is intuitively obvious that no solution can be found, and in practice, the rod area using the SDEP1 formulation would increase without bound. The failure of SDEP1 to find a solution is a direct result of the requirements for existence of an equilibrium point. One requirement is the boundedness of the feasible region of the design space. This requirement can be satisfied by arbitrarily imposing a maximum area constraint on the rod. However, at the feasible point where both the maximum area and the maximum stress constraints are active, it is not possible to change the rod area to move into the interior of the feasible region, violating the constraint qualification required for existence of an equilibrium point. Thus, the failure of SDEP1 (and fully stressed design) to find a solution to this simple problem is clarified by the EP existence theorem.

Because SDEP1 is a form of fully stressed design, it shares the advantages and disadvantages of fully stressed design. The primary advantage is the simple nature of a structural-sizing subproblem that requires no derivatives and is easily decomposed into a set of independent elemental problems. The disadvantages are: 1) there is no mechanism to ensure satisfaction of the

necessary conditions represented by statement (6) so that the resulting equilibrium point may not be an optimal point; and 2) constraints that have no explicit dependence on the sizing variables, such as displacement constraints, are not considered. An EP formulation is desired which satisfies the necessary conditions represented by statement (6) at an equilibrium point, and can satisfy displacement constraints. The two formulations described subsequently overcome these disadvantages of SDEP1.

**Structural-Design Formulation SDEP2.** As discussed for SDEP1, if EP subproblems  $i = 1, \dots, M$  are identified as structural-response subproblems each represented by statement (3), and subproblem 0 is identified with a structural-sizing subproblem given by statement (5), the constraint qualification relations (10) cannot be satisfied for any displacement constraint because the displacements would be fixed parameters within the structural-sizing subproblem. In addition, the constraint qualification relations may not be satisfied for certain stress and buckling constraints. Utilizing the third decomposition method described previously can overcome this difficulty.

In EP formulation SDEP2, the displacements  $\mathbf{u}^i$ , which are parameters independent of the sizing variables in structural-sizing constraint functions  $\mathbf{g}(\mathbf{v}, \mathbf{u}^i)$ , are replaced with a first-order Taylor series approximation given by

$$\mathbf{u}^{A^i}(\mathbf{v}, \mathbf{x}^i) \equiv \frac{d\mathbf{u}^i}{d\mathbf{v}}(\mathbf{v} - \mathbf{v}_i) + \mathbf{u}^i \quad (14)$$

In equation (14), the matrix  $d\mathbf{u}^i/d\mathbf{v}$  is a matrix of optimal sensitivity derivatives (ref. 8) of the displacements in subproblem  $i$  with respect to  $\mathbf{v}$ , and  $\mathbf{v}_i$  is the value of  $\mathbf{v}$  utilized in subproblem  $i$  when the sensitivities are calculated. All the design variables for structural-response subproblem  $i$  are utilized in equation (14) since  $\mathbf{x}^i \equiv (\mathbf{u}^i, d\mathbf{u}^i/d\mathbf{v}, \mathbf{v}_i)$ . This approximation satisfies the following two properties. First, the approximation depends explicitly on  $\mathbf{v}$  so that constraint qualification relations (10) necessary for equilibrium point existence can be satisfied. Second, it satisfies the conditions:  $\mathbf{u}^{A^i} = \mathbf{u}^i$  and  $\partial \mathbf{u}^{A^i}/\partial \mathbf{v} = \partial \mathbf{u}^i/\partial \mathbf{v}$  at the equilibrium point  $\mathbf{x} = \mathbf{x}^*$  where  $\mathbf{v} = \mathbf{v}_i = \mathbf{v}^*$ . This second set of conditions ensures the optimality of the design given by the equilibrium point because the EP necessary conditions represented by statement (9) are then the same as the NLP necessary conditions represented by statement (6).

Using the definition of equation (14), the structural-

sizing subproblem is given by the following statement:

$$\begin{aligned} \min_{\mathbf{x}^0 \equiv \mathbf{v}} W(\mathbf{v}) \\ \mathbf{g}_s(\mathbf{v}) \geq 0 \\ \mathbf{g}_u^i(\mathbf{u}^{A^i}(\mathbf{v}, \mathbf{x}^i)) \geq 0 \\ \mathbf{g}_\sigma^i(\mathbf{v}, \sigma_u^i(\mathbf{v}, \mathbf{u}^{A^i}(\mathbf{v}, \mathbf{x}^i))) \geq 0 \end{aligned} \quad (15)$$

The structural-response subproblems for SDEP2 are the subproblems represented by unconstrained minimizations given by statement (3) (or the necessary conditions given by equation (4)), and the following equations that determine the behavior variables  $d\mathbf{u}^i/d\mathbf{v}$  and  $\mathbf{v}_i$ :

$$\begin{aligned} \mathbf{K}(\mathbf{v}) \frac{d\mathbf{u}^i}{d\mathbf{v}} + \frac{\partial \mathbf{K}(\mathbf{v})}{\partial \mathbf{v}} \mathbf{u}^i = [0] \\ \mathbf{v}_i = \mathbf{v} \end{aligned} \quad (16)$$

for  $i = 1, \dots, M$ . The first of the two equations in (16) can be viewed as an optimal sensitivity analysis of  $\mathbf{u}^i$  with respect to parameters  $\mathbf{v}$  since  $\mathbf{u}^i$  are design variables of the NLP problem given by statement (3), and the second equation is a trivial identity which saves the values of the structural-sizing variables at which the sensitivity analysis was performed.

SDEP2 is equivalent to a NLP approach to structural design with a first-order Taylor series for an approximate analysis when the equilibrium point is found by the following sequential steps: 1) solve the subproblems represented by statement (3) (or necessary conditions represented by statement (4)) for  $\mathbf{u}^i$  for each  $i = 1, \dots, M$ ; 2) solve equations (16) for  $d\mathbf{u}^i/d\mathbf{v}$  and  $\mathbf{v}_i$ ; 3) use the quantities found in steps 1 and 2 in equation (14) to form approximate models for displacements  $\mathbf{u}^{A^i}(\mathbf{v}, \mathbf{x}^i)$ ; 4) utilize the approximate models of step 3 in the structural-sizing subproblem represented by statement (15) to solve for  $\mathbf{v}$ ; and 5) repeat steps 1 through 4 until the changes in the solutions from the previous iteration are smaller than a specified tolerance.

**Structural-Design Formulation SDEP3.** In the preceding decompositions derived within the EP framework, all the design constraints are included in the structural-sizing subproblem. In the SDEP3 decomposition, the displacement constraints  $\mathbf{g}_u^i$  are applied within structural-response subproblem  $i$ , so that the unconstrained structural-response subproblem, represented by statement (3), becomes a constrained minimization. This step corresponds to the first decomposition method described previously. However, the constrained structural-response subproblem no longer gives the response of the structure to the applied nodal forces  $\mathbf{F}_E^i$  because "virtual" nodal forces are also applied to satisfy the displacement constraints. These virtual nodal forces and the steps required to obtain the correct structural response are discussed subsequently.

The constrained minimization solved within structural-response subproblem  $i$  is

$$\min_{\mathbf{u}^i} \left( \frac{1}{2} (\mathbf{u}^i)^T \mathbf{K}(\mathbf{v}) \mathbf{u}^i - (\mathbf{F}_E^i)^T \mathbf{u}^i \right) \quad (17)$$

$$\mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) \geq 0$$

and the necessary conditions for this subproblem are

$$\mathbf{K}(\mathbf{v}) \mathbf{u}^i - \mathbf{F}_E^i - \left( \frac{\partial \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)}{\partial \mathbf{u}^i} \right)^T \boldsymbol{\lambda}^i = 0$$

$$\mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) \geq 0 \quad (18)$$

$$\boldsymbol{\lambda}^i \geq 0$$

$$(\boldsymbol{\lambda}^i)^T \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) = 0$$

It is apparent from the necessary conditions represented by statement (18) that quantities  $(\partial \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) / \partial \mathbf{u}^i)^T \boldsymbol{\lambda}^i$  are similar to the applied nodal forces  $\mathbf{F}_E^i$ , and can be considered as "virtual" nodal forces. If the Lagrange multipliers  $\boldsymbol{\lambda}^i$  in these virtual nodal forces can be forced to zero by modification of the sizing variables, then the correct structural response is obtained. To accomplish this task, the values of the Lagrange multipliers and their sensitivity derivatives with respect to the sizing variables must be supplied to the structural-sizing subproblem. Since statement (18) implies that the Lagrange multipliers and their sensitivity derivatives are zero for inactive constraints, and the sensitivity derivatives of the active displacement constraints are also zero, the sensitivity derivative equations to be solved in structural-response subproblem  $i$  are given by reference 8 as

$$\frac{\partial^2 L^i(\mathbf{u}^i, \mathbf{v}, \boldsymbol{\lambda}^i)}{\partial (\mathbf{u}^i)^T \partial \mathbf{u}^i} \frac{d\mathbf{u}^i}{d\mathbf{v}} - \left( \frac{\partial \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)}{\partial \mathbf{u}^i} \right)^T \frac{d\boldsymbol{\lambda}^i}{d\mathbf{v}} + \frac{\partial^2 L^i(\mathbf{u}^i, \mathbf{v}, \boldsymbol{\lambda}^i)}{\partial (\mathbf{u}^i)^T \partial \mathbf{v}} = [0] \quad (19)$$

$$\frac{\partial \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)}{\partial \mathbf{u}^i} \frac{d\mathbf{u}^i}{d\mathbf{v}} = [0]$$

In equations (19), the vector  $\mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)$  represents the active displacement constraints, the vector  $\boldsymbol{\lambda}^i$  are the Lagrange multipliers corresponding to these active displacement constraints, and the function  $L^i(\mathbf{u}^i, \mathbf{v}, \boldsymbol{\lambda}^i)$  is the Lagrangian function defined by

$$L^i(\mathbf{u}^i, \mathbf{v}, \boldsymbol{\lambda}^i) = \frac{1}{2} (\mathbf{u}^i)^T \mathbf{K}(\mathbf{v}) \mathbf{u}^i - (\mathbf{F}_E^i)^T \mathbf{u}^i - (\boldsymbol{\lambda}^i)^T \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) \quad (20)$$

$$= \frac{1}{2} (\mathbf{u}^i)^T \mathbf{K}(\mathbf{v}) \mathbf{u}^i - (\mathbf{F}_E^i)^T \mathbf{u}^i - (\boldsymbol{\lambda}^i)^T \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)$$

If the stress and buckling constraints are included within the structural-sizing subproblem, and they are calculated from stress resultants  $\mathbf{N}^i$  of structural-response subproblem  $i$ , the second decomposition method previously described is also utilized in SDEP3. As discussed for the SDEP1 formulation, this approach may not yield an optimum design for a redundant structure. However, substituting approximate models, which depend explicitly on the sizing variables, for the stress resultants (as in reference 9, or equivalently, in the third method for decomposition) yields an optimal EP formulation. Thus, the stress resultants, and their sensitivity derivatives with respect to the sizing variables are required, and are obtained from the equations

$$\mathbf{N}^i - \mathbf{N}^i(\mathbf{v}, \mathbf{u}^i) = 0 \quad (21)$$

$$\frac{d\mathbf{N}^i}{d\mathbf{v}} - \frac{\partial \mathbf{N}^i(\mathbf{v}, \mathbf{u}^i)}{\partial \mathbf{u}^i} \frac{d\mathbf{u}^i}{d\mathbf{v}} - \frac{\partial \mathbf{N}^i(\mathbf{v}, \mathbf{u}^i)}{\partial \mathbf{v}} = [0]$$

In summary, structural-response subproblem  $i$  is represented by statements (17) or (18), and statements (19) and (21) with the addition of the trivial conditions  $\mathbf{v}_i = \mathbf{v}$ , as in method SDEP2, and the equations

$$\mathbf{g}_{\mathbf{u}}^i = \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i) \quad (22)$$

$$\frac{d\mathbf{g}_{\mathbf{u}}^i}{d\mathbf{v}} = \frac{\partial \mathbf{g}_{\mathbf{u}}^i(\mathbf{u}^i)}{\partial \mathbf{u}^i} \frac{d\mathbf{u}^i}{d\mathbf{v}}$$

The design variables for this structural-response subproblem are  $\mathbf{x}^i \equiv (\mathbf{u}^i, \mathbf{N}^i, \boldsymbol{\lambda}^i, \mathbf{g}_{\mathbf{u}}^i, \mathbf{v}_i, d\mathbf{u}^i/d\mathbf{v}, d\mathbf{N}^i/d\mathbf{v}, d\boldsymbol{\lambda}^i/d\mathbf{v}, d\mathbf{g}_{\mathbf{u}}^i/d\mathbf{v})$ .

The formulation of the structural-sizing subproblem is now discussed. As mentioned previously, if the stress and buckling constraints are included within the structural-sizing subproblem and they are calculated from stress resultants  $\mathbf{N}^i$  of structural-response subproblem  $i$ , then the substitution of approximate models which depend explicitly on the sizing variables for the stress resultants yields an optimal EP formulation. In SDEP3, a first-order Taylor series approximate model for stress resultant  $\mathbf{N}^i$  is formed as

$$\mathbf{N}^i(\mathbf{v}, \mathbf{x}^i) \equiv \frac{d\mathbf{N}^i}{d\mathbf{v}} (\mathbf{v} - \mathbf{v}_i) + \mathbf{N}^i \quad (23)$$

where the matrix  $d\mathbf{N}^i/d\mathbf{v}$ , defined previously, can be viewed as the total derivative of the stress resultants with respect to  $\mathbf{v}$ . The quantities  $d\mathbf{N}^i/d\mathbf{v}$ ,  $\mathbf{v}_i$ , and  $\mathbf{N}^i$  are determined within structural-response subproblem  $i$  and are only parameters within the structural-sizing subproblem.

Although the displacement constraints are satisfied within the structural-response subproblems, the Lagrange multipliers corresponding to the active displacement constraints must be zero for the correct structural

response to be obtained. If a displacement constraint is not active within a structural-response subproblem, a provision within the structural-sizing subproblem is desired to guard against violation of the constraint. A constraint function that accomplishes this task within the structural-sizing subproblem is a first-order approximation in  $\mathbf{v}$  to the expression  $\mathbf{g}_u^i - \lambda^i$ . Component  $j$  of this approximation is given by:

$$g_{uj}^{Ai}(\mathbf{v}, \mathbf{x}^i) = \begin{cases} g_{uj}^i + \frac{dg_{uj}^i}{d\mathbf{v}}(\mathbf{v} - \mathbf{v}_i) & (g_{uj}^i > 0) \\ -\lambda_j^i - \frac{d\lambda_j^i}{d\mathbf{v}}(\mathbf{v} - \mathbf{v}_i) & (g_{uj}^i = 0) \end{cases} \quad (24)$$

Thus, if a displacement constraint is not active in structural-response subproblem  $i$ , the constraint function in equation (24) is a fully linearized version of the corresponding form in the SDEP2 formulation. And, if a displacement constraint is active in structural-response subproblem  $i$ , then a structural-sizing constraint with the functional form of equation (24) will coerce the virtual force in the structural-response subproblem to zero. The constraint function in equation (24) satisfies all the requirements of the EP existence theorem described previously except for the continuity requirement. Continuity, and therefore equilibrium point existence, could be established if the function in statement (24) is replaced with one which differs from it only over the small transition interval  $0 < g_{uj}^i < \epsilon$ . In this interval, the constant and gradient terms in equation (24) are replaced with continuous transition functions that range between  $-\lambda_j^i$  and  $g_{uj}^i$ , and between  $-d\lambda_j^i/d\mathbf{v}$  and  $dg_{uj}^i/d\mathbf{v}$ , respectively. This continuous form of the constraint function satisfies the existence requirement, but has not been found necessary in the computational procedure. Therefore, the form of statement (24) is retained.

With the previously mentioned definitions, the structural-sizing subproblem in SDEP3 becomes

$$\begin{aligned} \min_{\mathbf{x}^0 \equiv \mathbf{v}} \quad & W(\mathbf{v}) \\ & \mathbf{g}_s(\mathbf{v}) \geq 0 \\ & \mathbf{g}_\sigma^i(\mathbf{v}, \sigma_N(\mathbf{v}, \mathbf{N}^{Ai}(\mathbf{v}, \mathbf{x}^i))) \geq 0 \\ & \mathbf{g}_u^{Ai}(\mathbf{v}, \mathbf{x}^i) \geq 0 \end{aligned} \quad (25)$$

where  $i = 1, \dots, M$ , the stress and buckling constraints depend on the stress resultants as in the subproblem represented by statement (12), and the vector of "displacement" constraint functions  $\mathbf{g}^{Ai}(\mathbf{v}, \mathbf{x}^i)$  has elements given by equation (24). A proof of the optimality of the equilibrium point obtained from SDEP3 is given in the appendix.

## Implementation of SDEP algorithms

For practical structural-design problems, an equilibrium-programming-based solution algorithm should utilize existing finite element structural analysis software. Existing finite element programs can easily solve equations (4) for structural response, but are not well suited to solving a constrained NLP problem as represented by statement (17) (or equivalently, the necessary conditions as represented by statement (18)). Because of this limitation, the class of displacement constraints considered in the present implementations of solution algorithms is restricted to simple displacement limits such as  $\mathbf{g}_u^i(\mathbf{u}^i) = \mathbf{u}_{\max} - \mathbf{u}^i$ . In the following sections, solution procedures are described for solving the SDEP2 and SDEP3 structural-response, and structural-sizing subproblems. These procedures are used with the Engineering Analysis Language (EAL) structural analysis code (ref. 10) to solve some sample design problems. The sequential solution approach described previously, in which the structural-sizing subproblem and the structural-response subproblems are alternately solved, is the method chosen for the present study.

**Structural-Response Subproblems.** The formulation of the structural-response subproblem in SDEP2 uses standard methods for computing the structural response and the structural-response sensitivity derivatives (see ref. 11). Thus, existing finite element codes are well-suited to the structural-response subproblem formulation of SDEP2. However, formulation SDEP3 specifies inequality displacement constraints in the structural-response subproblems represented by statement (17). The restriction of the displacement constraints to simple bounds on displacements in the present study allows for an easily implemented active constraint set approach to the structural-response subproblems in SDEP3. In this approach, a structural-response subproblem is solved in an iterative manner so that a solution to the necessary conditions represented by statement (18) is obtained from a sequence of solutions to equation (4) that differ by the boundary conditions assumed when factoring the stiffness matrix  $\mathbf{K}$ . Specifically, when a displacement resulting from the solution to equation (4) violates either its maximum or minimum bound, the boundary conditions for the structural-response subproblem are modified to have that displacement specified to be the displacement limit when factoring  $\mathbf{K}$  for the next solution to equation (4). The static reactions corresponding to these specified displacements are computed by the finite element code. These static reactions equal the displacement-constraint Lagrange multipliers of statement (18) if the specified displacement is a lower bound, or the negative of the displacement-constraint Lagrange multipliers if the specified displacement is an upper bound (see

the appendix). If any Lagrange multiplier determined from these static reactions is negative, the corresponding specified displacement should be "freed" in factoring the stiffness matrix for the next solution to equation (4). Likewise, if any displacement exceeds its allowable bounds, the appropriate displacement bound is utilized within the boundary conditions in the next iteration as previously described. The correct active set of displacement constraints is determined, and the correct solution to the necessary conditions represented by statement (18) are obtained when no calculated displacement violates its bounds, and no Lagrange multiplier is negative. An additional advantage of this active constraint set approach to solving the SDEP3 structural-response subproblems is that after  $u^i$  and  $\lambda^i$  are determined, the sensitivity derivatives  $du^i/dv$  and  $d\lambda^i/dv$  are determined by standard structural-response sensitivity analysis methods, and it is not necessary to solve equation (19). However, solving a constrained structural-response subproblem is typically more expensive than solving an unconstrained one because of the necessity of refactoring the stiffness matrix and resolving the system of equations for every change in the assumed set of active constraints. For the present study, the EAL sensitivity derivative runstreams of reference 12 were modified to generate sensitivity derivative information for both SDEP2 and SDEP3.

**Structural-Sizing Subproblem.** In the present study, the structural-sizing NLP subproblems represented by statements (15) for SDEP2 and (25) for SDEP3 are replaced with linear programming approximations to these subproblems having move limits as additional side constraints on the sizing variables. Move limits are easily incorporated within the EP theory given in reference 6, so the details are omitted here. However, the method used for move limit control in the present study differs from reference 6 in that the move limit side constraints are modified before each solution of the structural-sizing subproblem, not after convergence to the equilibrium point corresponding to those move limits. The move limits chosen allow a 30% relative change in the design variables initially. The relative change allowed is reduced by 10% before each solution to the structural-sizing subproblem, but it is not reduced below 6%. This strategy for controlling the move limits is fairly crude and has been found to be too restrictive for some of the example problems. In cases for which a linear programming approximation to a structural-sizing subproblem has no feasible solution, the objective function is modified to include a constraint violation penalty using the method described in reference 11. Thus, the subproblem will essentially minimize the worst constraint violation when no feasible solution exists. Also, because the stress constraints are linearized in this approach, there is, in theory, no dif-

ference between the stress constraints calculated from either displacements or stress resultants. The subroutine DDLPRS of reference 13 was incorporated within EAL to solve the linear programming problems.

For a large structural model, there can be thousands of elements that must be examined for local stress and buckling constraints, and must have the optimal structural sizes determined. To reduce the number of design variables and constraints in the structural-sizing subproblem, a large finite element model can be partitioned into regions. Within each region, all the elements are sized by a few design variables. In addition, the local constraints within each region can be "lumped" using the Kresselmeier-Steinhauser cumulative constraint described in reference 14. These reduction methods are utilized only for the civil transport wing problem described subsequently in the results section of the present paper.

## Results

Two sample problems, a simple ten-bar truss example and a complex high-speed transport wing example, are evaluated using the SDEP2 and SDEP3 methods described above. SDEP2 is designated a "conventional" method herein, since it is equivalent to conventional NLP formulations using a Taylor series approximation for displacements for evaluating the constraints. In the present study, only a single load condition is considered in the structural response. The results presented consider minimum gauge and displacement constraints only for the ten-bar truss example problem, and minimum gauge, displacement, local buckling, and stress constraints for the high-speed transport wing example problem.

### Ten-Bar Truss

The minimum weight ten-bar truss problem is illustrated in figure 3. The vertical and horizontal members are each 360 in. long. The material properties assumed are those for aluminum with a Young's modulus of  $10^7$  psi, a Poisson's ratio of 0.3, and a density of  $0.1 \text{ lb}_m/\text{in}^3$ . Two 100,000 lb loads are applied, and the upper displacement limits  $\delta_1$ ,  $\delta_2$ ,  $\delta_3$ , and  $\delta_4$  for the displacement constraints are shown in the figure. Results for two design cases are presented. In case 1, the limits for the displacement constraints are  $\delta_1 = \delta_4 = 2.0$  in. No  $\delta_2$  or  $\delta_3$  limits are assumed for this case. In case 2, the assumed displacement limits are  $\delta_1 = 2.0$  in.,  $\delta_2 = 0.75$  in.,  $\delta_3 = 0.5$  in., and  $\delta_4 = 1.8$  in. No stress or local buckling constraints are considered in either case. The bar cross-sectional areas are the design variables for the structural-sizing subproblem. The minimum cross-sectional area allowed is  $0.1 \text{ in}^2$ . For both cases, results are obtained using two values for

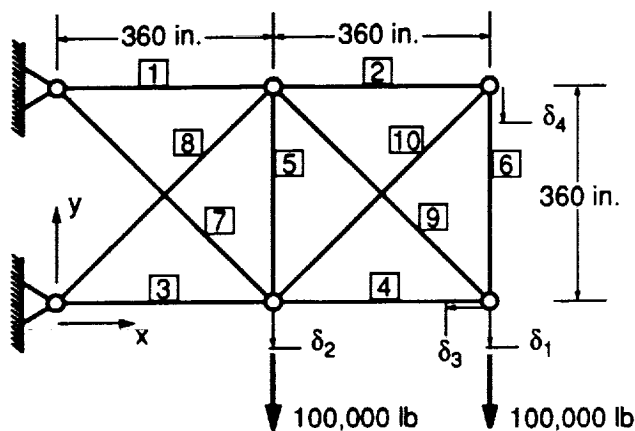


Figure 3. Schematic diagram of geometry, loads, and displacement limits for ten-bar truss example problem.

Table 1. Final bar cross-sectional areas and weight for ten-bar truss, case 1, initial area = 25 in<sup>2</sup>

Bar Number	Ref. 11 Area, in <sup>2</sup>	SDEP2 Area, in <sup>2</sup>	SDEP3 Area, in <sup>2</sup>
1	30.52	30.95	31.05
2	0.10	0.10	0.10
3	23.20	23.26	23.23
4	15.22	14.73	14.77
5	0.10	0.10	0.10
6	0.55	0.10	0.10
7	7.46	8.63	8.63
8	21.04	20.78	20.69
9	21.53	20.95	21.02
10	0.10	0.10	0.10
Weight, lb <sub>m</sub>	5060.9	5062.1	5064.7

the initial cross-sectional area of all members, 10.0 in<sup>2</sup> and 25.0 in<sup>2</sup>. In the solution process, eighty steps of the sequential iterative solution method using the move limit strategy described previously are utilized for both the SDEP2 and SDEP3 formulations.

**Design case 1 results.** The iteration histories of the truss weight resulting from formulations SDEP2 (i.e., the "conventional" formulation) and SDEP3 for the case having two displacement constraints are shown in figure 4 for the 25 in<sup>2</sup> initial cross-sectional area. In this figure and in subsequent figures, the value of the ordinate is the value at the beginning of the iteration indicated on the abscissa; therefore the initial design conditions are denoted as the conditions at iteration 1. The results for weight in figure 4 are very similar for

the two SDEP formulations. The iteration histories for the individual design variables also show little difference between the two formulations. The slow convergence of the weight is due to the restrictive move limit strategy utilized in the present study. The move limits control how fast the areas of bars 2, 5, 6, and 10 can be reduced, and do not allow these areas to achieve minimum gauge until iteration 63. Good agreement of final results from the two SDEP formulations with those reported in reference 11 is indicated by the final bar cross-sectional areas and truss weights given in Table 1. Because of small oscillations in the iteration history results, the SDEP3 results in this table are for the next-to-last iteration to present SDEP2 and SDEP3 results that are "in phase". The agreement between the results from the two EP formulations is slightly better than their agreement with reference 11, but overall, the differences are minor.

The iteration histories of the truss weights for this design case with the 10 in<sup>2</sup> initial cross-sectional area are shown in figure 5. The structural-sizing subproblems for both SDEP2 and SDEP3 have no feasible solutions for the initial two iterations (i.e., for results corresponding to abscissa values 2 and 3 in figure 5). In these iterations, the structural-sizing subproblem minimizes the infeasibility, and both methods give identical results since they are at the side-constraint boundaries determined by the move limits. The results from the two SDEP methods differ for the next few iterations but again agree well at the later iterations. Again, the slow convergence to the optimum design is due to the restrictive move limit strategy.

**Design case 2 results.** The iteration histories of the truss weights resulting from formulations SDEP2 and SDEP3 for the design case having four displacement constraints are shown in figure 6 for the 25 in<sup>2</sup> initial cross-sectional area. Again, there is little difference between the weight histories of the two methods shown in figure 6, or in the individual design variable histories. However, much larger differences in the iteration histories of weight occur for the 10 in<sup>2</sup> initial cross-sectional area. The results shown in figure 7, indicate that weight decreases for the initial iteration of SDEP3 while the weight increases for SDEP2. This initial decrease in weight for SDEP3 adversely affects the convergence of the method, and is discussed in detail subsequently. A closer examination of the results from structural-sizing subproblems for SDEP2 and SDEP3 reveals that the structural-sizing subproblem for SDEP2 has no feasible solution for the initial three iterations, but the structural-sizing subproblem for SDEP3 has no feasible solution for the initial eight iterations. For these iterations, the structural-sizing subproblem minimizes the infeasibility. For the 10 in<sup>2</sup> initial cross-sectional area, all the displacement constraints are initially active for

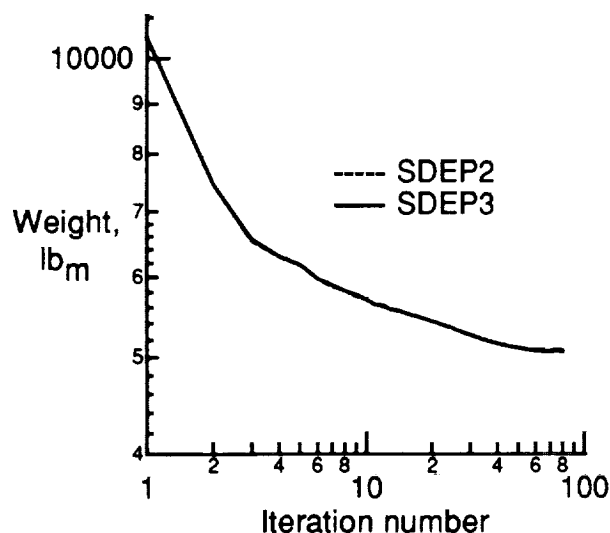


Figure 4. Convergence histories for weight of ten-bar truss example problem, design case 1 with initial area for sizing variables of 25 in<sup>2</sup>.

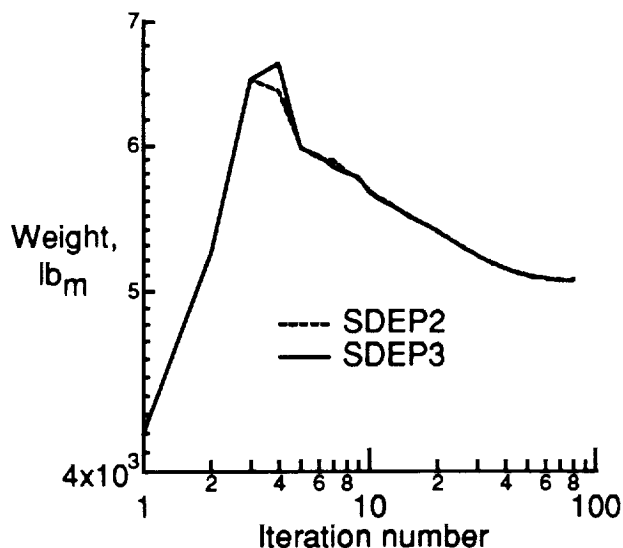


Figure 5. Convergence histories for weight of ten-bar truss example problem, design case 1 with initial area for sizing variables of 10 in<sup>2</sup>.

the SDEP2 formulation, while only the displacements limited by  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$  are initially active in the SDEP3 formulation with corresponding virtual forces in the 6,000 - 85,000 lb range. For these infeasible iterations, to reduce the structural-sizing-subproblem objective function of weight plus a penalty proportional to the largest virtual force (which corresponds to the  $\delta_2$  displacement limit), the initial changes to the sizing variables reduce all the member areas except for bars 7 and 8. Decreasing these member areas decreases the stiffness of the structure and reduces the virtual force required to satisfy the  $\delta_2$  displacement limit. No feasible solution is found in the next few iterations of the SDEP3 structural-sizing subproblem, but in these iterations more constraints contribute to the penalty and weight increases as more sizing variables begin to move toward the optimal values. The iteration histories for cross-sectional areas of bars 1, 4, and 6 are shown in figure 8 to illustrate typical sizing variable results. The significantly different sizing variable results in the initial iterations combined with the restrictive move limit control strategy severely reduces the speed of convergence of the SDEP3 method in these results.

### High-Speed Civil Transport Wing

The second sample problem considered in this study is the structural sizing of the wing for a proposed high-speed civil transport concept described in reference 15. The details defining this structural-design problem are too numerous to list so only a summary of its features is presented. The finite element model used for the

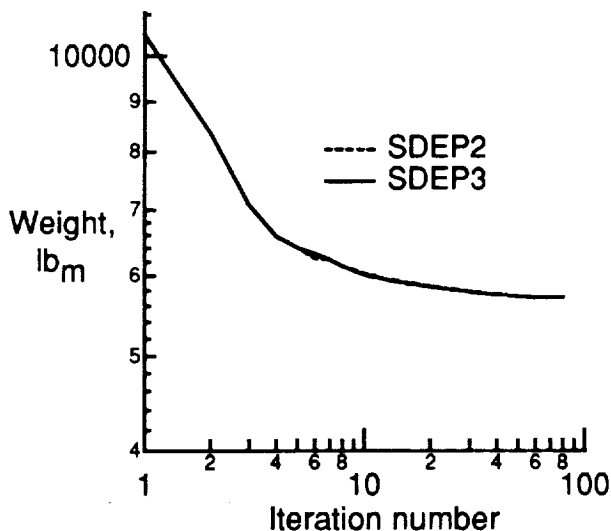


Figure 6. Convergence histories for weight of ten-bar truss example problem, design case 2 with initial area for sizing variables of 25 in<sup>2</sup>.

structural-response subproblem is shown in figure 9. The upper wing cover panels are removed in this figure to illustrate the rib and spar web arrangement. The cover panels are titanium honeycomb sandwich panels, and the shear webs are titanium sine-wave webs. The model is relatively detailed with 1728 nodes (10,144 degrees of freedom) and 2447 elements. A single load condition is analyzed in the structural response, a 2.5g

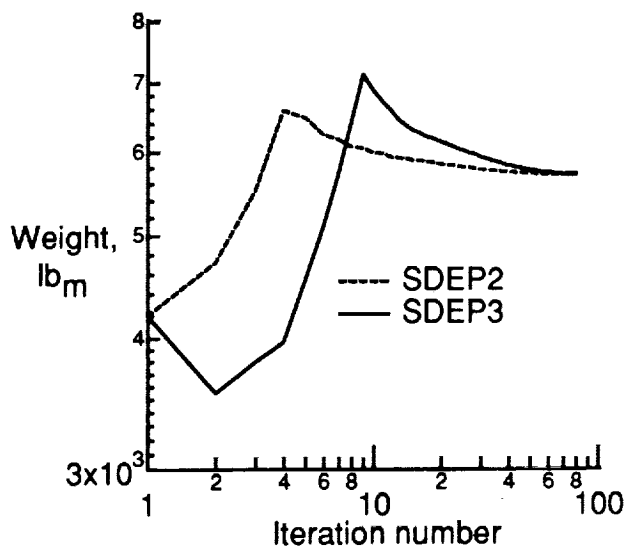


Figure 7. Convergence histories for weight of ten-bar truss example problem, design case 2 with initial area for sizing variables of  $10 \text{ in}^2$ .

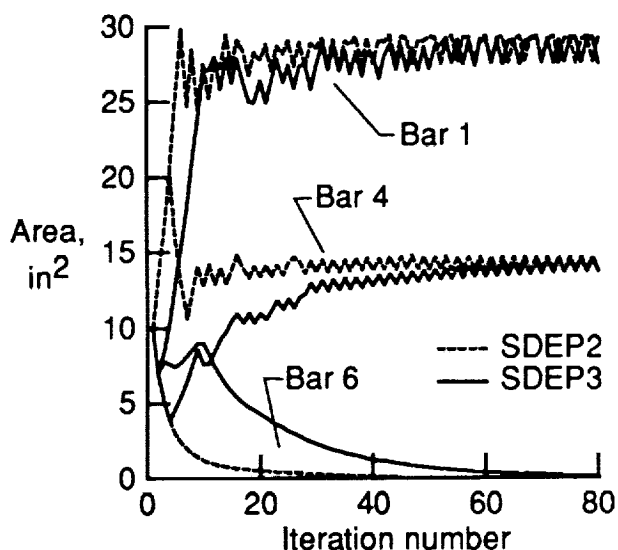


Figure 8. Convergence histories for selected bar areas of ten-bar truss example problem, design case 2 with initial area for sizing variables of  $10 \text{ in}^2$ .

balanced, symmetric supersonic pull-up maneuver.

There are 41 sizing variables considered in the structural-sizing subproblem. These design variables include honeycomb sandwich facesheet thicknesses, honeycomb sandwich core heights, and sine-wave web gauges. The model is partitioned so that each sizing variable sizes multiple elements. The constraints considered are minimum and maximum gauges for the

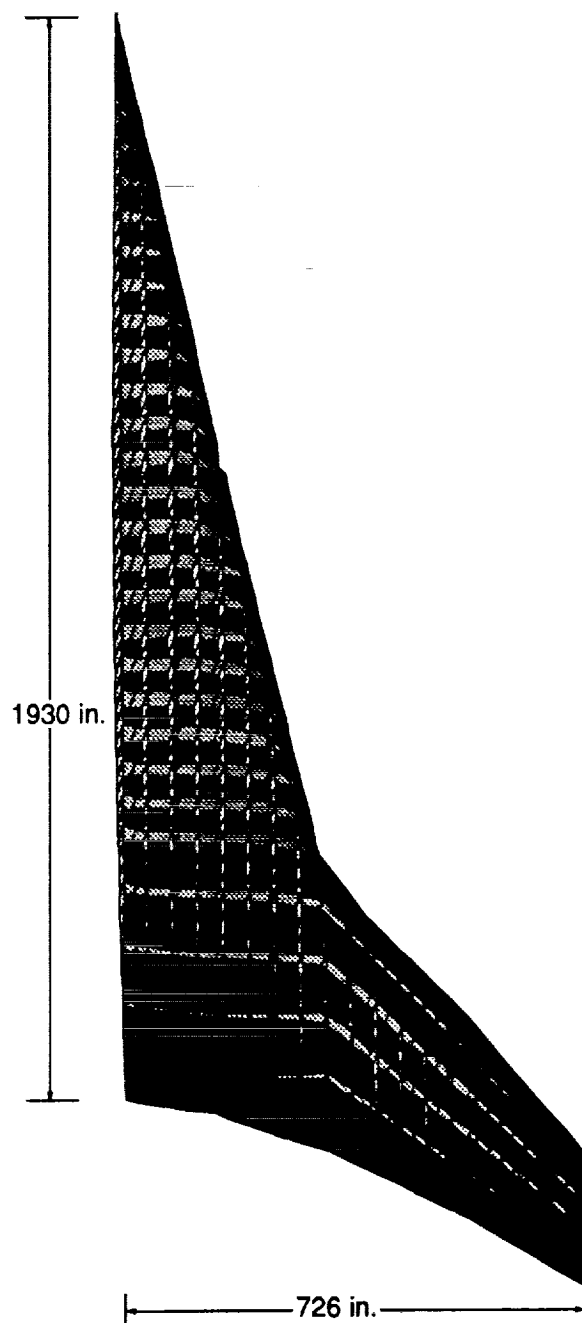


Figure 9. Finite element model of high-speed civil transport wing with upper cover panels removed.

facesheets and webs, minimum and maximum thicknesses for the honeycomb core, maximum element stresses, local buckling due to compression and shear, and a 12-foot limit on maximum deflection for the  $2.5g$  maneuver. The constraint "lumping" method using the Kresselmeier-Steinhauser function described previously is utilized to reduce the large number of possible stress



and local buckling constraints from several thousand to only 106. The value of the Kresselmeier-Steinhauser parameter used in the present study is 50.

Because of the computational expense of the detailed structural model, the two equilibrium programming formulations were executed for only 16 iterations. The relative move limit for the last iteration was thus slightly greater than the 6% minimum value. The history of structural weight for the SDEP2 and SDEP3 methods is shown in figure 10. The results from the two methods are nearly indistinguishable, and the weight reaches its limiting value of about 41,800 lb<sub>m</sub> by the eleventh iteration. The individual constraint having the largest effect on the structural weight is the 12-foot displacement limit. The convergence history of this constraint for the SDEP2 and SDEP3 design methods is shown in figure 11. The scales for the displacement constraint in SDEP2 and the corresponding virtual force constraint in SDEP3 are chosen to show equivalent violation at the beginning of iteration 3 because at this iteration both methods have the same set of sizing variables. The figure shows that the virtual force constraint in SDEP3 and the normalized displacement constraint in SDEP2 follow similar paths to reach feasible design space, however although both constraints are related to the wing displacement limit, they cannot be directly compared. The two methods yield differences in the design history for some of the wing sizing variables as illustrated in figure 12. Although the weight histories shown previously are nearly identical, noticeable differences in the sizing variable histories for the two methods occur. Sizing variables 24 and 25 correspond to thicknesses of shear webs with the normalized ordinate value 1.0 corresponding to a thickness of 0.1 in. Sizing variables 18 and 34 are typical, normalized honeycomb core and facesheet thicknesses, with normalizing factors of 5.0 in. and 0.2 in., respectively. Although some differences appear in the sizing variable histories of the two SDEP methods with the third iteration, the differences essentially disappear by the last iteration.

### Concluding Remarks

The present study is an initial investigation in the use of equilibrium programming (EP) as a framework for design decomposition. The advantages of performing decomposition within the EP framework are the availability of existence theorems and optimality criteria for the solution to the decomposed problem. Three decompositions of structural-design problems into structural-sizing and structural-response subproblems are developed using this equilibrium programming framework. Two of the methods developed (i.e., SDEP1 and SDEP2) are shown to be equivalent to the existing conventional methods of fully stressed design, and of nonlinear programming using approximation

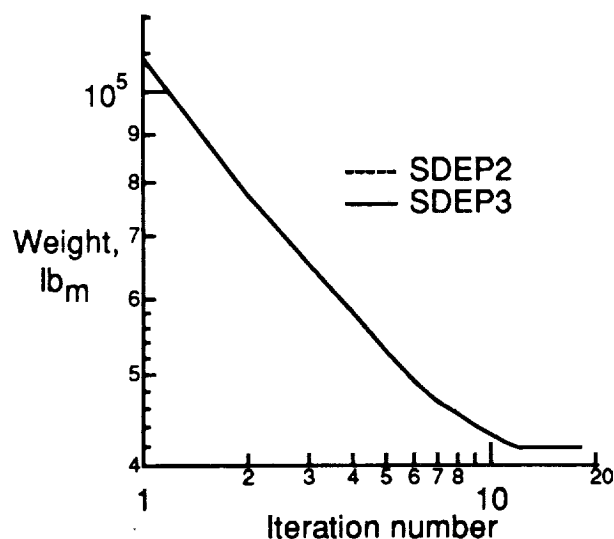


Figure 10. Convergence histories for weight of high-speed civil transport wing example problem.

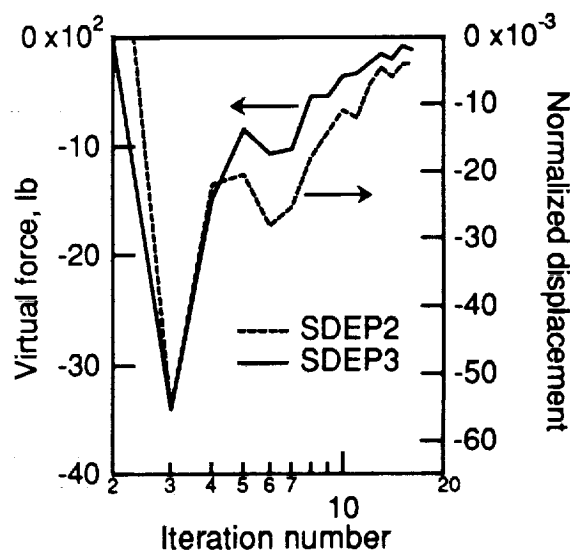


Figure 11. Convergence histories for displacement constraint of high-speed civil transport wing example problem.

concepts for rapid analysis. In the third method for optimal structural design presented (i.e., SDEP3), the displacement constraints are applied within the structural-response subproblems. However, the structural-sizing subproblem acquires new constraints which effectively control the magnitude of "virtual" forces that are applied in the structural-response subproblems to satisfy the displacement constraints. The optimality of solutions generated by this third method is demon-

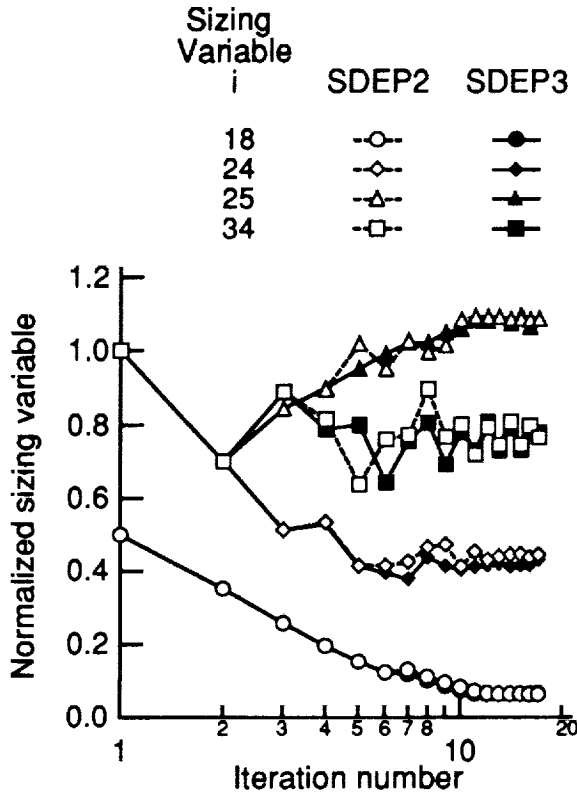


Figure 12. Convergence histories for selected sizing variables of high-speed civil transport wing example problem.

strated analytically, and by numerical examples of a simple ten-bar truss design problem, and of a complex, high-speed civil transport wing design problem. In the example problems, the number of iterations to convergence for the third design method are approximately equal to the conventional nonlinear programming approach using first-order Taylor series approximations for a rapid analysis. However, the total computational expense is slightly larger because of the active constraint set approach to solving the constrained structural-response subproblems. In addition, the convergence of the third method sometimes degrades when iterations occur which have no feasible solution to the structural-sizing subproblem. This degradation appears related to the tendency of the solution algorithm for the structural-sizing subproblem to minimize the magnitude of the "virtual" force constraint violation by increasing the flexibility of the structure.

## Appendix

The equivalence of the necessary conditions for the SDEP3 equilibrium programming method to a nonlinear programming approach that yields an optimal structural design is demonstrated in this appendix. For

simplicity, only displacement constraints, and a single structural response corresponding to one load case are considered. Also, the displacement constraints are assumed to have the form

$$\mathbf{g}_u = \mathbf{u}_{\max} - \mathbf{u} \geq 0 \quad (A1)$$

where it is also assumed that the nodal reference frames are oriented so that all deflection limits are maximums.

The nonlinear programming formulation, such as given by statement (5), simplifies to:

$$\begin{aligned} \min_{\mathbf{v}} W(\mathbf{v}) \\ \mathbf{g}_u = \mathbf{u}_{\max} - \mathbf{u}(\mathbf{v}) \geq 0 \end{aligned} \quad (A2)$$

where  $\mathbf{u}(\mathbf{v})$  is the solution to the structural-response problem given by statement (3) with necessary conditions given by equation (4). In this nonlinear programming approach, the necessary conditions for statement (A2) are given by statement (6), two conditions of which simplify to

$$\begin{aligned} \frac{\partial W(\mathbf{v})}{\partial \mathbf{v}} + \lambda_*^T \frac{d\mathbf{u}_*}{d\mathbf{v}} &= 0 \\ \lambda_* &\geq 0 \end{aligned} \quad (A3)$$

where  $\mathbf{u}_*$  (and the corresponding  $\lambda_*$ ) indicates those displacements which are at their constraint boundaries in the problem represented by statement (A2). The term  $d\mathbf{u}_*/d\mathbf{v}$  in the problem represented by statement (A3) is determined by taking the derivative of equation (4) with respect to  $\mathbf{v}$

$$\mathbf{K}(\mathbf{v}) \frac{d\mathbf{u}}{d\mathbf{v}} + \frac{\partial \mathbf{K}(\mathbf{v}) \mathbf{u}}{\partial \mathbf{v}} = [0] \quad (A4)$$

and forming the matrix  $d\mathbf{u}_*/d\mathbf{v}$  from terms of the matrix  $d\mathbf{u}/d\mathbf{v}$  corresponding to the displacements at their constraint boundaries.

In the SDEP3 equilibrium programming formulation of this problem, only those displacements at the constraint boundary effectively contribute to the necessary conditions so the structural-sizing subproblem given by statement (25) simplifies to:

$$\begin{aligned} \min_{\mathbf{x}^0 \equiv \mathbf{v}} W(\mathbf{v}) \\ -\frac{d\lambda_*^1}{d\mathbf{v}} (\mathbf{v} - \mathbf{v}_1) - \lambda_*^1 \geq 0 \end{aligned} \quad (A5)$$

and the structural-response subproblem given by state-

ments (17) and (19) simplifies to:

$$\begin{aligned} \min_{\mathbf{x}^1} \quad & \left( \frac{1}{2} (\mathbf{u}^1)^T \mathbf{K}(\mathbf{v}) \mathbf{u}^1 - \mathbf{F}_E^T \mathbf{u}^1 \right) \\ & \mathbf{u}_{\max} - \mathbf{u}^1 \geq 0 \\ & \mathbf{v}_1 - \mathbf{v} = 0 \end{aligned} \quad (A6)$$

$$\mathbf{K}(\mathbf{v}) \frac{d\mathbf{u}^1}{d\mathbf{v}} + \begin{bmatrix} [0] \\ \frac{d\lambda_*^1}{d\mathbf{v}} \end{bmatrix} + \frac{\partial \mathbf{K}(\mathbf{v}) \mathbf{u}^1}{\partial \mathbf{v}} = [0]$$

$$\frac{d\mathbf{u}_*^1}{d\mathbf{v}} = [0]$$

where the formal definition of  $\mathbf{x}^1$  is  $\mathbf{x}^1 \equiv (\mathbf{u}^1, \lambda_*^1, \mathbf{v}_1, d\mathbf{u}^1/d\mathbf{v}, d\lambda_*^1/d\mathbf{v})$ ,  $\mathbf{u}^1$  is ordered into those displacements not at the constraint boundary  $\mathbf{u}_*^1$ , followed by those at the constraint boundary so that  $\mathbf{u}^1 = (\mathbf{u}_*^1, \mathbf{u}_*^1)$ , and the quantities  $\lambda_*^1$  are the Lagrange multipliers for the displacements  $\mathbf{u}_*^1$  at the constraint boundaries. The two relations of interest in the necessary conditions for subproblem (A5) becomes

$$\begin{aligned} \frac{\partial W(\mathbf{v})}{\partial \mathbf{v}} + \mu^T \frac{d\lambda_*^1}{d\mathbf{v}} &= 0 \\ \mu &\geq 0 \end{aligned} \quad (A7)$$

where  $\mu$  are the Lagrange multipliers for the inequality constraints in statement (A5).

The desired result is to show that if  $\mathbf{x}^0 = \mathbf{v}$  and  $\mathbf{x}^1$  is an equilibrium point for SDEP3 (i.e., solves (A5) and (A6) and thus their necessary conditions), then  $\mathbf{v}$  and  $\mathbf{u} = \mathbf{u}^1$  satisfy the necessary conditions for an optimal solution of the nonlinear programming formulation, in particular equation (4) and statement (A3). To accomplish this result, express the two necessary conditions for the structural-response subproblem represented by statement (A6) as

$$\begin{aligned} \mathbf{K}(\mathbf{v}) \mathbf{u}^1 + \begin{bmatrix} 0 \\ \lambda_*^1 \end{bmatrix} - \mathbf{F}_E &= 0 \\ \lambda_*^1 &\geq 0 \end{aligned} \quad (A8)$$

where the Lagrange multipliers corresponding to unconstrained displacements are explicitly shown as zero. Comparing necessary conditions represented by statements (4) and (A8), it is apparent that  $\mathbf{u}$  cannot equal  $\mathbf{u}^1$  unless  $\lambda_*^1 = 0$ . This condition is assured by the second relation in statement (A8), and the inequality constraints in the structural-sizing subproblem represented by statement (A5). Combining equation (A4), the last two equations in statement (A6), and the con-

dition  $\mathbf{u} = \mathbf{u}^1$ , yields the equation

$$\begin{bmatrix} \mathbf{K}_A & | & \mathbf{K}_B \\ \hline \mathbf{K}_B^T & | & \mathbf{K}_D \end{bmatrix} \left( \begin{bmatrix} \frac{d\mathbf{u}_u}{d\mathbf{v}} \\ \frac{d\mathbf{u}_*}{d\mathbf{v}} \end{bmatrix} - \begin{bmatrix} \frac{d\mathbf{u}_*^1}{d\mathbf{v}} \\ [0] \end{bmatrix} \right) = \begin{bmatrix} [0] \\ \frac{d\lambda_*^1}{d\mathbf{v}} \end{bmatrix} \quad (A9)$$

where the matrix  $\mathbf{K}$  has been partitioned as shown. The first set of equations in (A9) can be solved to give

$$\frac{d\mathbf{u}_u^1}{d\mathbf{v}} = \frac{d\mathbf{u}_u}{d\mathbf{v}} + \mathbf{K}_A^{-1} \mathbf{K}_B \frac{d\mathbf{u}_*}{d\mathbf{v}} \quad (A10)$$

Substituting equations (A10) in the second set of equations in (A9) results in

$$(\mathbf{K}_D - \mathbf{K}_B^T \mathbf{K}_A^{-1} \mathbf{K}_B) \frac{d\mathbf{u}_*}{d\mathbf{v}} = \frac{d\lambda_*^1}{d\mathbf{v}} \quad (A11)$$

Finally, substituting equation (A11) into the necessary conditions represented by statement (A7), the desired result of the necessary conditions represented by statement (A3) is obtained with

$$\lambda_*^T = \mu^T (\mathbf{K}_D - \mathbf{K}_B^T \mathbf{K}_A^{-1} \mathbf{K}_B) \quad (A12)$$

With the demonstration that the equilibrium point of SDEP3 satisfies equation (4) and the first relation in (A3), the final step required is proof that the inequality in (A3) is also satisfied. To do so requires the assumption that the SDEP3 equilibrium point is isolated (i.e., there is no other equilibrium point within an  $\epsilon$  neighborhood of  $\mathbf{v}$ ). The proof will be by contradiction. First assume that at the equilibrium point of a SDEP3 problem, one or more components of  $\lambda_*$  computed from equation (A12) are negative. This assumption implies the existence of a more optimal, neighboring, feasible point. Specifically, within an  $\epsilon$  neighborhood of  $\mathbf{v}$  and  $\mathbf{u}$ , quantities  $\mathbf{v}_\epsilon$  and  $\mathbf{u}_\epsilon$  exist and satisfy equation (4), the inequality constraint in statement (A2), and the relation  $W(\mathbf{v}_\epsilon) < W(\mathbf{v})$ . Since  $\mathbf{u}_\epsilon$  is the solution to an unconstrained potential energy minimization (i.e., equation (4)) with the stiffness matrix  $\mathbf{K}(\mathbf{v}_\epsilon)$ , and it also satisfies the displacement constraints in statement (A2), then  $\mathbf{u}^1 = \mathbf{u}_\epsilon$  is the solution to the constrained potential energy minimization given in statement (A6) for the same stiffness matrix. Thus, the necessary condition equations in both statements (4) and (A8) must be satisfied, and  $\lambda_*^1 = 0$  for both equations to hold. Finally, the choice  $\mathbf{x}^0 = \mathbf{v}_1 = \mathbf{v}_\epsilon$  satisfies the inequality constraint in statement (A5) with the weight  $W$  in statement (A5) smaller for  $\mathbf{x}^0 = \mathbf{v}_\epsilon$  than for  $\mathbf{x}^0 = \mathbf{v}$ . Since the  $\epsilon$  chosen is arbitrary, either the original point is not an equilibrium point or the equilibrium point

is not isolated, contradicting the original assumptions. Thus, the inequality in statement (A3) must hold, and the equilibrium point is also an optimal point.

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### References

1. Schmit, L. A., and Miura, H.: *Approximation Concepts for Efficient Structural Synthesis*. NASA CR-2552, March 1976.
2. Barthelemy, J.-F. M.: Engineering Design Applications of Multilevel Optimization Methods. *Computer Aided Optimum Design of Structures: Applications*, Proceeding of the first International Conference, Southampton, UK, June 1989, pp. 113-122.
3. Jollissaint, C. H., Arvanitidis, N. V., and Luenberger, D. G.: Decompositions of Real and Reactive Power Flows: a Method Suited for On-Line Applications. *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-91, no. 2, March-April 1972, pp. 661-670.
4. Garcia, C. B., and Zangwill, W. I.: *Pathways to Solutions, Fixed Points, and Equilibria*. Prentice-Hall, 1981.
5. Lansing, W., Dwyer, W., Emerton, R., and Ranalli, E.: Application of Fully Stressed Design Procedures to Wing and Empennage Structures. *Journal of Aircraft*, vol. 8, no. 9, 1971, pp. 683-688.
6. Zangwill, W. I., and Garcia, C. B.: Equilibrium Programming: The Path Following Approach and Dynamics. *Mathematical Programming*, vol. 21, 1981, pp. 262-289.
7. Bloebaum, C. L.: *Formal and Heuristic System Decomposition Methods in Multidisciplinary Synthesis*. NASA CR-4413, December, 1991.
8. Sobieszczanski-Sobieski, J., Barthelemy, J.-F. M., and Riley, K. M.: Sensitivity of Optimum Solutions to Problems Parameters. *AIAA Journal*, vol. 20, no. 9, September 1982, pp. 1291-1299.
9. Vanderplaats, G. N., and Salajegheh, E.: New Approximation Method for Stress Constraints in Structural Synthesis. *AIAA Journal*, vol. 27, no. 3, March, 1989, pp. 352-358.
10. Whetstone, W. D.: *EISI-EAL Engineering Analysis Language Reference Manual*, Engineering Information Systems, San Jose, CA., July, 1983.
11. Haftka, R. T., Gurdal, Z., and Kamat, M. P.: *Elements of Structural Optimization*, Kluwer Academic Publishers, Boston, MA., 1990.
12. Camarda, C. J., and Adelman, H. M.: *Static and Dynamic Structural-Sensitivity Derivative Calculations in the Finite-Element-Based Engineering Analysis Language (EAL) System*. NASA TM-85743, March, 1984.
13. Anonymous: *User's Manual IMSL Math/Library*. IMSL, Houston, TX., December, 1989.
14. Kreisselmeier, G., and Steinhauser, R.: Systematic Control Design by Optimizing a Vector Performance Index. *Proceedings of IFAC Symposium on Computer-Aided Design of Control Systems*, Zurich, 1979, pp. 113-117.
15. Robins, A. W., Dollyhigh, S. M., Beissner, F. L., Geiselhart, K., Martin, G. L., Shields, E. W., Swanson, E. E., Coen, P. G., and Morris, S. J.: *Concept Development of a Mach 3.0 High-Speed Civil Transport*. NASA TM-4058, September, 1988.



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